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ROBUSTNESS OF THE ONE-SAMPLE WILCOXON SIGNED-RANK TEST WHEN INDEPENDENCE IS NOT ASSUMED*

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1. THE WILCOXON SIGNED-RANK TEST

1.1 THE ONE-SAMPLE LOCATION PROBLEM

The *location parameter* is a number obtained from a probability distribution which indicates where the distribution is "centered" or "located". Let this number be denoted by θ. The one-sample location problem is the problem of testing the hypothesis,

$$H_o: \theta = \theta_o$$

against either the one sided alternative

$$H_1: \theta < \theta_o \quad (\text{or } \theta > \theta_o)$$

or the two-sided alternative

$$H_1: \theta \neq \theta_o.$$

This will be solved without making any assumptions about the specific form or parameter values of the underlying population distribution (which is the case for *distribution-free* statistical procedures, one of which is the test under consideration). Instead the following basic assumptions are set down:

1. The set $(X_1\ X_2, \ldots, X_n)$ constitute the sample drawn; the random variables are independent and have the same distribution.
2. The population distribution is continuous.
3. The population distribution is symmetric.

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* This is taken from the author's M. Sc. unpublished thesis "The Power and Robustness of the One-Sample Wilcoxon Signed-Rank Test", submitted to U.P. Statistical Center, 1972.

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Under the well-defined assumed distribution of the classical theory, the expected value is the location parameter. In the more general class of distributions satisfying the assumptions given above, the expected value when existing is of little use because generally its sampling distribution depends on the (unknown) population distribution. However, for any distribution, the *median* m always exists as that number for which

$$Pr\ (X > m) = 0.50.$$

Hence, in this instance, the median m is considered as the location parameter.

1.2 THE TEST STATISTIC

A random sample of n observations is drawn from a continuous and symmetric population with median m. Under H_0, x_i is symmetric about m_0; hence the differences

$$D_i = X_i - m_0 \qquad i = 1,2,\ldots,n$$

are symmetrically distributed about 0. This means that positive and negative differences of equal absolute magnitude have the same probability of occurence; i.e. if c is any positive number then,

$$Pr\ (D_i \geq -c) = Pr\ (D_i \geq c) = 1 - Pr\ (D_i < c).$$

If no reference to a specific random variable X_i is made, the deviation $X - m_0$ from the hypothesized median will be denoted by D.

It was previously mentioned that if H_0 is true the probability distribution of D is symmetric about 0. Under H_1 this is not the case. Nevertheless, whatever is the value of the population median m, the following relationship concerning the cumulative distribution function of D will hold true,

$$F_{|D|}(u) = F_D(u) - F_D(u) \text{ for every } u \geq 0.$$

The absolute differences $|D_1|, |D_2|, \ldots, |D_n|$ are ordered from smallest to largest by assigning them ranks $1,2,\ldots,n$ at the same time keeping track of the signs of the differences D_i. After this is done, there will be a set of n ranks and a corresponding set of n plus and minus signs. The rank i is associated with a plus or minus sign according to the sign of $D_j = X_j - m_o$ where the rank of $|D_j|$ (denoted by $r(|D_j|)$) is i.

Let a sign indicator function be defined as follows,

$$Z_{(i)} = \begin{cases} 1 & \text{if } D_j > 0 \text{ and } r(|D_j|) = i, \\ o & \text{if } D_j \leq 0 \text{ and } r(|D_j|) = i. \end{cases}$$

Then the Wilcoxon Signed-Rank Test (hereafter written as WSRT) statistic can be defined as

$$T = \sum_{i=1}^{n} iZ_{(i)} \qquad (1)$$

It has been seen from equation (1) that T is the sum of the ranks of the absolute values $|D_j|$ corresponding to differences $X_j - m_o > o$. The random variables $Z_{(i)}$ are independent Bernouilli variables with parameter p_i which are not identical under H_1.

Let $|D|_{(i)}$ denote the ith order statistic among $|D_1|$, $|D_2|$, $\ldots |D_n|$. The parameter p_i is then defined as,

$$\begin{aligned} p_i &= Pr(Z_{(i)} = 1) \\ &= Pr(|D|)_{(i)} \text{ is such that } D > 0. \end{aligned} \qquad (2)$$

Utilizing the expression for the distribution of the ith order statistic, the distribution of $|D|_{(i)}$ can be written as

$$f_{|D|_{(i)}}(u) = [n! \,/\, (i\text{-}1)!\,(n\text{-}i)!]\, [F_{|D|}(u)]^{i-1}\, [1\text{-}F_{|D|}(u)]^{n-i}\, f_{|D|}(u)$$

This marginal density can be used to derive an expression for p_i as defined in equation (2). In its final form, this expression is as follows,

$$p_i = n\binom{n-1}{i-1} \int_0^{\infty} [F_D(u) - F_D(-u)]^{i-1}\, [1 - F_D(u) + F_D(-u)]^{n-i} f_D(u)\,du.$$

If H_o is true P_i is evaluated when X is symmetric about m_o, i.e. when D is symmetric about 0. Under this condition,

$$F_D(-u) = 1 - F_D(u). \qquad (4)$$

Substituting (4) in (3),

$$p_i = n \binom{n-1}{i-1} \int_0^{\infty} [2F(u) - 1]^{i-1} \; [2 - 2F(u)]^{n-i} f(u)du. \quad (5)$$
$$D$$

Applying the transformation $v = 2F(u) - 1$, (5) becomes,

$$p_i = n/2 \binom{n-1}{i-1} \int_0^1 v^{i-1} (1-v)^{n-i} dv \quad (6)$$

The integral in equation (6) is just the beta function,

$$B(i, n-i+1) = (i-1)! \; (n-i+1-1)! \; / \; (i+n-i+1-1)!$$

$$= (i-1)! \; (n-i)! \; / \; n! \quad (7)$$

From (6) and (7),

$$p_i = n/2 \binom{n-1}{i-1} (i-1)! \; (n-i)! \; / \; n!$$

$$= 1/2.$$

Therefore under H_o,

$$P_i = Pr(Z_{(i)} = 1) = 1/2, \qquad i = 1,2,\ldots,n. \quad (8)$$

This means that if the null hypothesis is true, the event $X_j - m_o > 0$ and $X_j - m_o < 0$ where $|X_j - m_o| = |D|_{(i)}$ are equally likely. This fact is the basis for formulating the probability distribution of T under H_o.

From equation (1) it can be seen that T is completely determined by the sign indicators $Z_{(i)}$. This if the statistic T has a given value t (where $t = 0,1,\ldots,n(n+1)/2$) then it is completely defined by the set of n-tuples,

$$A = \left\{ (Z_{(1)}, Z_{(2)}, \ldots, Z_{(n)}) \; ; \; Z \in \{0,1\} \; , \; \sum_{i=1}^{n} iZ_{(i)} = t \right\},$$

such that

$$Pr(T = t) = Pr(A).$$

Let n(t) be the number of sample points (n-tuples) in A (i.e. the number of elements in the set A such that $T = t$). Suppose that for one of these sample points, exactly k specified ranks $r_1, r_2, \ldots, r_k$ correspond to positive signs of $X - m_o$. The probability of this particular sample point would then be equal to,

$$\prod_{i=1}^{k} p_{r_i} \prod_{j \neq r_i} (1-p_j).$$

Since there are n(t) sample points in A,

$$\Pr(A) = \sum_{L=1}^{n(t)} \left[\prod_{i=1}^{k} p_{r_i} \prod_{j \neq r_i} (1-p_j) \right]_L$$

$$= \Pr(T = t). \qquad (9)$$

For example, in a random sample of size n,

$$T = 3 \text{ if and only if } A = \left\{ (1,1,0,\ldots,0),\ (0,0,1,\ldots,0) \right\}$$

Therefore from (9),

$$\Pr(T = 3) = \Pr(1,1,0,\ldots,0) + \Pr(0,0,1,0,\ldots,0)$$

$$= p_1 p_2 \prod_{i=3}^{n} (1-p_i) + (1-p_1)(1-p_2) p_3 \prod_{i=4}^{n} (1-p_i),$$

where, if H_o is not true, the value of p_i can be obtained from equation (3) provided the population distribution function is known.

Equation (9) gives the probability distribution of T for any value of the median m. If $m = m_o$,

$$\prod_{i=1}^{k} p_{r_i} \prod_{j \neq r_i} (1-p_j) = \underbrace{(1/2.1/2...1/2)}_{k \text{ times}} \underbrace{(1/2.1/2...1/2)}_{n-k \text{ times}}$$

$$= (1/2^k) \quad (1/2^{n-k})$$

$$= 1/2^n.$$

Hence equation (9) becomes,

$$\Pr(T = t)_{H_0} = \underbrace{1/2^n + 1/2^n + \dots + 1/2^n}_{n(t) \text{ terms}}$$

$$= n(t)/2^n. \qquad (10)$$

Equation (10) gives the distribution of T under H_0. Its mean and variance are evaluated to be,

$$(11)$$

$$E(T)_{H_0} = \sum_{i=1}^{n} iE(Z_{(i)})_{H_0} = \sum_{i=1}^{n} i(1/2) = 1/2 \sum_{i=1}^{n} i$$

$$= n(n+1) / 4;$$

$$Var(T)_{H_0} = \sum_{i=1}^{n} i^2 Var(Z_{(i)})_{H_0} = \sum_{i=1}^{n} i^2 (1/2) \quad (1-1/2)$$

$$= 1/4 \sum_{i=1}^{n} i^2$$

$$= n(n+1) \ (2n+1) / 24 \qquad (12)$$

It is easy to generate the distribution of T under H_0 for a given sample size n. The extreme values of T are 0 n(n+1)/2. For each vector $(Z_{(1)}, Z_{(2)}, \dots, Z_{(n)})$ associated with a given value of T, the assignment of signs has a *conjugate* assignment which results when the plus and minus signs are interchanged. The value of T for this conjugate assignment is,

$$t_{conj} = \sum_{i=1}^{n} i(1-Z_{(i)}) = n(n+1) / 2 - \sum_{i=1}^{n} iZ_{(i)}$$

$$= n(n+1) / 2 - t. \qquad (13)$$

Obviously likewise,

$$Pr(T = t) = Pr(T = t_{conj}) \qquad (14)$$

From (13) and (14) it can be shown that the distribution of T under H_0 is symmetric about its mean. A condition for symmetry about the mean for a discrete random variable X is that,

$$Pr(X = x) = Pr(X = 2E(X) - x) \text{ for all } x > E(X).$$

This is satisfied by T since for all $t > n(n+1) / 4$,

$$\begin{aligned} Pr(T = 2n(n+1) / 4 - t) &= Pr(T = n(n+1 / 2 - t) \\ &= Pr(T = t_{conj}) \\ &= Pr(T = t). \end{aligned}$$

Because of this symmetry property, only one-half of the distribution under H_0 need be determined. The example below illustrates this.

In a random sample of size n = 4, T can assume values which vary from 0 to 4(5) / 2. It is symmetric about its mean 4(5) / 4. Considering the upper half of the set of T values first construct the following table:

T = t	Ranks associated with positive differences	Number n(t) of sample points
10	(1,2,3,4)	1
9	(2,3,4)	1
8	(1,3,4)	1
7	(1,2,4,), (3,4)	2
6	(1,2,3), (2,4)	2
5	(1,4), (2,3)	2

(Note: A similar table can be made for the lower half of the set of T values.)

Using the data above and the fact that $2^n = 2^4 = 16$, the distribution of T under H_o is obtained to be,

$$Pr(T = t) = \begin{cases} 1/16, & \text{if } t = 0,1,2,8,9,10; \\ 2/16, & \text{if } t = 3,4,5,6,7; \\ 0 & \text{otherwise.} \end{cases}$$

This procedure can be applied for any sample size n. For large n however, generating the probability distribution becomes a tendious process. There are prepared tables for the sampling distribution of T under H_o one of which is reproduced here (see Table I).

1.3 REJECTION REGIONS

For a preassigned level of significance P(I) the critical region can be set up. However distribution-free statistics are discrete random variables. This is the case particularly with the WSRT statistic. Therefore it is not possible to choose just any number between 0 and 1 to designate the value of P(I) since the possible P(I) values are confined to the jump joints in the cumulative distribution of the test statistic. The procedure adopted is to define the rejection region in such a way that an *exact P(I)* is the largest number which does not exceed the preassigned level of significance (henceforth also to be denoted as the *nominal P(I)*.

Suppose H_1: $m > m_o$. As the number of random variables in the sample greater than the hypothesized median increases, the value of the statistic T increases. This follows from the definition of T since the number of absolute differences corresponding to positive signs likewise increases. Hence an appropriate rejection region for this alternative given a nominal P(I) is,

$$R = (T;\ T \geq t_{p(I)}),$$

where $t_{tp(I)}$ is the critical T value.

Consider H_1: $m < m_o$. As the number of observations in the sample less than the hypothesized median increases, the value of T decreases since more of the absolute differences would correspond to negative signs. The rejection region for a given nominal P(I) is thus,

$$R = (T;\ T \leq t_{p(1)}).$$

Since the sampling distribution of T is symmetric, the rejection region for the two-sided alternative H_1: $m \neq m_o$ for a given nominal P(I) is,

$$R = (T;\ T \leq t_{p(1)}/_2 \text{ or } T \geq t_{p(1)}/_2).$$

The critical T values are obtained from the probability distribution of T under H_o given in Table I. The table is cumulative from each extreme to the mean but not beyond. Consider the case of n = 3 for which the frequency distribution of T is the following:

T = t	n(t)
0	1
1	1
2	1
3	2
4	1
5	1
6	1

The relationship between the above and the tabular values (see Table I) is,

T = t	P	
0	0.125 = 1/8	$Pr(T \leq t)$
1	0.250 = (1+1)1/8	
2	0.375 = (1+1+1)/8	
3	0.625 = (1+1+1+2)/8	$Pr(T \leq t) = Pr(T \geq t)$
4	0.375 = (1+1+1)/8	$Pr(T \geq t)$
5	0.250 = (1+1)/8	
6	0.125 = (1/8	

The range of Table I is $3 \leq n \leq 15$. Suppose it is given that the nominal P(I) is 0.50 for a one-sided test against the alternative H_1: $m < m_o$. Required the critical value of T. From the column of cumulative probabilities P the largest number which does not exceed 0.50 is 0.375. This is the exact P(I).

Hence the critical value $t_{p(1)}$ such that

$$Pr(T \leq t_{p(1)}) = 0.375$$

is 2.

The application of the WSRT is illustrated with a test at a significance level 0.10 of the null hypothesis, H_0: m = 2 versus the alternative, H_1: m $\neq$ 2. The random sample consists of the following n = 7 values;

-3, -7, 1, 9, 4, 10, 12.

From the sample, the following tabulation can be made:

X_i	$D_i = X_i - 2$	$\lvert D_i \rvert$	$r(\lvert D_i \rvert)$
-3	-5	5	3
-7	-9	9	6
1	-1	1	1
9	7	7	4
4	2	2	2
10	8	8	5
12	10	10	7

The above tabulation shows that r ($\lvert D_i \rvert$) = 2, 4, 5, and 7 correspond to positive differences and hence,

$$Z_{(2)} = Z_{(4)} = Z_{(5)} = Z_{(7)} = 1 \text{ while } Z_{(1)} = Z_{(3)} = Z_{(6)} = 0.$$

Therefore the observed value for T is,

$$T = \sum_{i=1}^{n} iZ_{(i)} = 1(0) + 2(1) + 3(0) + 4(1) + 5(1) + 6(0) + 7(1) = 16.$$

From Table I the two-sided rejection region corresponding to the given nominal P(I) is,

$$(T;\ T \leq 3 \text{ or } T \geq 25)$$

with exact P(I) equal to 2(0.039) or 0.078. Since the observe T value lies outside of the rejection region, the conclusion would be to reject H_o.

In this study, the robustness of the WSRT against a certain departure from the assumption of independent observations is investigated. In the consideration of the problem, a Monte Carlo simulation was performed.

TABLE I: Wilcoxon Signed-Rank Distributions[1]

Sample Size (n)													
3		4		5		6		7		8		9	
T	P	T	P	T	P	T	P	T	P	T	P	T	P
0	.125	0	.062	0	.031	0	.016	0	.008	0	.004	0	.002
1	.250	1	.125	1	.062	1	.031	1	.016	1	.008	1	.004
2	.375	2	.188	2	.094	2	.047	2	.023	2	.012	2	.006
3	.625	3	.312	3	.156	3	.078	3	.039	3	.020	3	.010
		4	.438	4	.219	4	.109	4	.055	4	.027	4	.014
3	.625	5	.562	5	.312	5	.156	5	.078	5	.039	5	.020
4	.375			6	.406	6	.219	6	.109	6	.055	6	.027
5	.250	5	.562	7	.500	7	.281	7	.148	7	.074	7	.037
6	.125	6	.438			8	.344	8	.188	8	.098	8	.049
		7	.312	8	.500	9	.422	9	.234	9	.125	9	.064
		8	.188	9	.406	10	.500	10	.289	10	.156	10	.082
		9	.125	10	.312			11	.344	11	.191	11	.102
		10	.062	11	.219	11	.500	12	.406	12	.230	12	.125
				12	.156	12	.422	13	.469	13	.273	13	.150
				13	.094	13	.344	14	.531	14	.320	14	.180
				14	.062	14	.281			15	.371	15	.213
				15	.031	15	.219	14	.531	16	.422	16	.248
						16	.156	15	.469	17	.473	17	.285
						17	.109	16	.406	18	.527	18	.326
						18	.078	17	.344			19	.367
						19	.047	18	.289	18	.527	20	.410
						20	.031	19	.234	19	.473	21	.455
						21	.016	20	.188	20	.422	22	.500
								21	.148	21	.371		
								22	.109	22	.320	23	.500
								23	.078	23	.273	24	.455
								24	.055	24	.230	25	.410
								25	.039	25	.191	26	.367
								26	.023	26	.156	27	.326
								27	.016	27	.125	28	.285
								28	.008	28	.098	29	.248
										29	.074	30	.213
										30	.055	31	.180
										31	.039	32	.150
										32	.027	33	.125
										33	.020	34	.102
										34	.012	35	.082
										35	.008	36	.064
										36	.004	37	.049
												38	.037
												39	.027
												40	.020
												41	.014
												42	.010
												43	.006
												44	.004
												45	.002

[1] Reproduced from the book "A Nonparametric Introduction to Statistics" by Kraft and Van Eeden [14].

TABLE I: Wilcoxon Signed-Rank Distributions
(Lower Tail)

Sample Size (n)													
10		11		12		13		14		15			
T	P	T	P	T	P	T	P	T	P	T	P	T	P
0	.001	0	.000	0	.000	0	.000	0	.000	0	.000	31	.053
1	.002	1	.001	1	.000	1	.000	1	.000	1	.000	32	.060
2	.003	2	.001	2	.001	2	.000	2	.000	2	.000	33	.068
3	.005	3	.002	3	.001	3	.001	3	.000	3	.000	34	.076
4	.007	4	.003	4	.002	4	.001	4	.000	4	.000	35	.084
5	.010	5	.005	5	.002	5	.001	5	.001	5	.000	36	.094
6	.014	6	.007	6	.003	6	.002	6	.001	6	.000	37	.104
7	.019	7	.009	7	.005	7	.002	7	.001	7	.001	38	.115
8	.024	8	.012	8	.006	8	.003	8	.002	8	.001	39	.126
9	.032	9	.016	9	.008	9	.004	9	.002	9	.001	40	.138
10	.042	10	.021	10	.010	10	.005	10	.003	10	.001	41	.151
11	.053	11	.027	11	.013	11	.007	11	.003	11	.002	42	.165
12	.065	12	.034	12	.017	12	.009	12	.004	12	.002	43	.180
13	.080	13	.042	13	.021	13	.011	13	.005	13	.003	44	.195
14	.097	14	.051	14	.026	14	.013	14	.007	14	.003	45	.211
15	.116	15	.062	15	.032	15	.016	15	.008	15	.004	46	.227
16	.138	16	.074	16	.039	16	.020	16	.010	16	.005	47	.244
17	.161	17	.087	17	.046	17	.024	17	.012	17	.006	48	.262
18	.188	18	.103	18	.055	18	.029	18	.015	18	.008	49	.281
19	.216	19	.120	19	.065	19	.034	19	.018	19	.009	50	.300
20	.246	20	.139	20	.076	20	.040	20	.021	20	.011	51	.319
21	.278	21	.160	21	.088	21	.047	21	.025	21	.013	52	.339
22	.312	22	.183	22	.102	22	.055	22	.029	22	.015	53	.360
23	.348	23	.207	23	.117	23	.064	23	.034	23	.018	54	.381
24	.385	24	.232	24	.133	24	.073	24	.039	24	.021	55	.402
25	.423	25	.260	25	.151	25	.084	25	.045	25	.024	56	.423
26	.461	26	.289	26	.170	26	.095	26	.052	26	.028	57	.445
27	.500	27	.319	27	.190	27	.108	27	.059	27	.032	58	.467
		28	.350	28	.212	28	.122	28	.068	28	.036	59	.489
		29	.382	29	.235	29	.137	29	.077	29	.042	60	.511
		30	.416	30	.259	30	.153	30	.086	30	.047		
		31	.449	31	.285	31	.170	31	.097				
		32	.483	32	.311	32	.188	32	.108				
		33	.517	33	.339	33	.207	33	.121				
				34	.367	34	.227	34	.134				
				35	.396	35	.249	35	.148				
				36	.425	36	.271	36	.163				
				37	.455	37	.294	37	.179				
				38	.485	38	.318	38	.196				
				39	.515	39	.342	39	.213				
						40	.368	40	.232				
						41	.393	41	.251				
						42	.420	42	.271				
						43	.446	43	.292				
						44	.473	44	.313				
						45	.500	45	.335				
								46	.357				
								47	.380				
								48	.404				
								49	.428				
								50	.452				
								51	.476				
								52	.500				

TABLE I: Wilcoxon Signed-Rank Distributions
(Upper Tail)

Sample Size (n)													
10		11		12		13		14		15		16	
T	P	T	P	T	P	T	P	T	P	T	P	T	P
28	.500	33	.517	39	.515	46	.500	53	.500	60	.511	91	.042
29	.461	34	.483	40	.485	47	.473	54	.476	61	.489	92	.036
30	.423	35	.449	41	.455	48	.446	55	.452	62	.467	93	.032
31	.385	36	.416	42	.425	49	.420	56	.428	63	.445	94	.028
32	.348	37	.382	43	.396	50	.393	57	.404	64	.423	95	.024
33	.312	38	.350	44	.367	51	.368	58	.380	65	.402	96	.021
34	.278	39	.319	45	.339	52	.342	59	.357	66	.381	97	.018
35	.246	40	.289	46	.311	53	.318	60	.335	67	.360	98	.015
36	.216	41	.260	47	.285	54	.294	61	.313	68	.339	99	.013
37	.188	42	.232	48	.259	55	.271	62	.292	69	.319	100	.011
38	.161	43	.207	49	.235	56	.249	63	.271	70	.300	101	.009
39	.138	44	.183	50	.212	57	.227	64	.251	71	.281	102	.008
40	.116	45	.160	51	.190	58	.207	65	.232	72	.262	103	.006
41	.097	46	.139	52	.170	59	.188	66	.213	73	.244	104	.005
42	.080	47	.120	53	.151	60	.170	67	.196	74	.227	105	.004
43	.065	48	.103	54	.133	61	.153	68	.179	75	.211	106	.003
44	.053	49	.087	55	.117	62	.137	69	.163	76	.195	107	.003
45	.042	50	.074	56	.102	63	.122	70	.148	77	.180	108	.002
46	.032	51	.062	57	.088	64	.108	71	.134	78	.165	109	.002
47	.024	52	.051	58	.076	65	.095	72	.121	79	.151	110	.001
48	.019	53	.042	59	.065	66	.084	73	.108	80	.138	111	.001
49	.014	54	.034	60	.055	67	.073	74	.097	81	.126	112	.001
50	.010	55	.027	61	.046	68	.064	75	.086	82	.115	113	.001
51	.007	56	.021	62	.039	69	.055	76	.077	83	.104	114	.000
52	.005	57	.016	63	.032	70	.047	77	.068	84	.094	115	.000
53	.003	58	.012	64	.026	71	.040	78	.059	85	.084	116	.000
54	.002	59	.009	65	.021	72	.034	79	.052	86	.076	117	.000
55	.001	60	.007	66	.017	73	.029	80	.045	87	.068	118	.000
		61	.005	67	.013	74	.024	81	.039	88	.060	119	.000
		62	.003	68	.010	75	.020	82	.034	89	.053	120	.000
		63	.002	69	.008	76	.016	83	.029	90	.047		
		64	.001	70	.006	77	.013	84	.025				
		65	.001	71	.005	78	.011	85	.021				
		66	.000	72	.003	79	.009	86	.018				
				73	.002	80	.007	87	.015				
				74	.002	81	.005	88	.012				
				75	.001	82	.004	89	.010				
				76	.001	83	.003	90	.008				
				77	.000	84	.002	91	.007				
				78	.000	85	.002	92	.005				
						86	.001	93	.004				
						87	.001	94	.003				
						88	.001	95	.003				
						89	.000	96	.002				
						90	.000	97	.002				
						91	.000	98	.001				
								99	.001				
								100	.001				
								101	.000				
								102	.000				
								103	.000				
								104	.000				
								105	.000				

2. ROBUSTNESS

2.1 ROBUST TESTS

It is often that when statistical tests are applied, little is known of the validity of the assumptions. Before applying a given test it is imperative to recognize first any departure from assumptions that may be brought about by the actual situation. If such departure(s) exist, the next step step is to investigate whether the test is sensitive to it. This enables one to decide whether to proceed in using the test, modifying it, or substituting with another test whose performance is not considerably affected by the said departure.

The property which makes a test insensitive to changes in the underlying assumptions is called the *robustness* of the test. To identify a test as *robust* against a given departure from assumptions one must investigate the effect of the change on the performance of the test on the basis of at least one of the following:

1. the level of significance for a fixed rejection region;
2. the rejection region given for a fixed level of significance;
3. the power of the test;
4. the asymptotic relative efficiency with respect to other tests.

In dealing with the problem, this paper has adopted the first approach; namely, the investigation of the effect of the departure on the level of significance for fixed rejection regions.

2.2 MODEL

The departure from the assumption of independence being considered may be brought about by a situation wherein only a few (c) observations can be drawn per day, and where experiments have to be conducted for several (n) days to yield the needed number (nc) of observations. Observations on the same day might depend on a particular effect, the result of possible daily changes in the experimental conditions. The sample size is denoted by nc and the observations are grouped in n blocks with c observations per block. The possible change of conditions is introduced as a random block effect.

Let the random variables after blocking be denoted as X_{ij} $(i = 1,2,\ldots,n;\ j = 1,2,\ldots,c)$. Consider the model,

$$X_{ij} = U_i + V_{ij} \tag{15}$$

where $U_1, U_2, \ldots, U_n, V_{11}, V_{12}, \ldots, V_{nc}$ are assumed independent with distributions,

$$\Pr(U_i \leq u) = G(u) \qquad \text{(G and K are continuous and}$$
$$\Pr(V_{ij} \leq v) = K(v) \qquad \text{symmetric)},$$

and having parameters,

$$E(U_i) = m, \qquad E(V_{ij}) = 0,$$
$$Var\ (U_i) = J^2, \qquad Var(V_{ij}) = \sigma^2.$$

The random variables X_{ij} will therefore have the following parameters;

$$E(X_{ij}) = E(U_i + V_{ij}) = E(U_i) + E(V_{ij}) = m + 0 = m;$$

$$Var(X_{ij}) = Var(U_i + V_{ij}) = Var(U_i) + Var(V_{ij}) = J^2 + \sigma^2;$$

$$Cov(X_{ij}X_{i'j'}) = E(X_{ij}X_{i'j'}) - E(X_{i'j'})\ E(X_{i'j'})$$

$$= E(\ (U_i + V_{ij})\ (U_{i'} + V_{i'j'})\ ) - m^2$$

$$= E(U_iU_{i'} + V_{ij}U_{i'} + V_{i'j'}U_i + V_{ij}V_{i'j'}) - m^2$$

$$= E(U_iU_{i'}) + E(V_{ij})E(U_{i'}) + E(V_{ij})E(U_i) + E(V_{ij})E(V_{i'j'}) - m^2$$

$$= E(U_iU_{i'}) - m^2 \tag{16}$$

If $i \neq i'$,

$$E(U_iU_{i'}) = E(U_i)E(U_{i'}) = m^2,$$

and therefore equation (16) becomes,

$$Cov(X_{ij}X_{i'j'}) = m^2 - m^2 = 0. \tag{17}$$

If $i = i'$,

$$E(U_iU_{i'}) = E(U^2_i) = Var\ (U_i) + E^2(U_i)$$

$$= J^2 + m^2,$$

and hence (16) becomes,

$$Cov(X_{ij}X_{ij'}) = J^2 + m^2 - m^2 = J^2. \qquad (18)$$

The nature of the dependence can be seen from (17) and (18); two random variables in the same block are dependent while two random variables belonging to different blocks are independent.

2.3 EFFECT ON THE TYPE I PROBABILITY

Let the exact type I probability of the WSRT under the model be denoted as *exact P'(I)*. This will be compared with the exact P(I) if the underlying assumptions of section 1.1 are valid. The rejection regions considered are:

1. $\langle T;\ T = 0,1 \rangle$ (for exact $P(I) = 0.008$);
2. $\langle T;\ T = 0,1,2,3,4,5, \rangle$ (for exact $P(I) = 0.039$);
3. $\langle T;\ T = 0,1,2,3,4,5,6,7,8 \rangle$ (for exact $P(I) = 0.098$).

Given the null hypothesis H_o: $m = m_o$, let

$$|X_{ij} - m_o|_{(k)} \qquad (i = 1,2,\ldots,n;\ j = 1,2,\ldots,c),$$

be the kth order statistic among $|X_{11}\text{-}m_o|$, $|X_{12}\text{-}m_o|,\ldots,$ $|X_{nc}\text{-}m|$. To obtain the values of the exact P'(I) the probabilities under H_o,

$$p'_k = Pr(\ |X_{ij} - m_o|_{(k)} \text{ is such that } X_{ij} - m_o > 0) \quad (k = 1,2,\ldots,nc)$$

are needed. These probabilities were empirically generated by means of a Monte Carlo simulation performed with the aid of a computer[1]. The following steps were involved:

1. 150 samples of size 10 were drawn from a N(0,1) population (N(a,b) denotes a normal distribution with mean *a* and variance *b*). Let the random variables be denoted by

$$Y_1,\ Y_2,\ Y_3,\ Y_4,\ Y_5,\ Y_6,\ Y_7,\ Y_8,\ Y_9,\ Y_{10}.$$

2. In each sample let,

$$U_1 = (2)^{1/2} Y_1 + 1,$$
$$U_2 = (2)^{1/2} Y_2 + 1.$$

The random variables U_1 and U_2 are normally distributed with mean 1 and variance 2. Let,

$$Y_3 = V_{11}, \quad Y_7 = V_{21},$$
$$Y_4 = V_{12}, \quad Y_8 = V_{22},$$
$$Y_5 = V_{13}, \quad Y_9 = V_{23},$$
$$Y_6 = V_{14}, \quad Y_{10} = V_{24}.$$

3. In each sample define a new set of random variables in the following manner:

$$X_{11} = (2^{1/2} Y_1 + 1) + Y_3 = U_1 + V_{11},$$
$$X_{12} = (2^{1/2} Y_1 + 1) + Y_4 = U_1 + V_{12},$$
$$X_{13} = (2^{1/2} Y_1 + 1) + Y_5 = U_1 + V_{13},$$
$$X_{14} = (2^{1/2} Y_1 + 1) + Y_6 = U_1 + V_{14},$$
$$X_{21} = (2^{1/2} Y_2 + 1) + Y_7 = U_2 + V_{21},$$
$$X_{22} = (2^{1/2} Y_2 + 1) + Y_8 = U_2 + V_{22},$$
$$X_{23} = (2^{1/2} Y_2 + 1) + Y_9 = U_2 + V_{23},$$
$$X_{24} = (2^{1/2} Y_2 + 1) + Y_{10} = U_2 + V_{24}. \quad (19)$$

The random variables in (19) form a sample of size nc = 8, n = 2, c = 4, satisfying equation (15) with i = 1, 2, j = 1, 2, 3, 4, and where,

U_i is N(1,2) distributed,
V_{ij} is N(0,1) distributed,

---

1. A terminal at De La Salle College was used.

and the moments are,

$$E(X_{ij}) = 1,$$
$$Var(X_{ij}) = 3,$$
$$Cov(X_{ij}X_{i'j'}) = \begin{cases} 2, & i = i'; \\ 0 & i \neq i'. \end{cases}$$

4. In this simulation the null hypothesis is H_o: $m = 1$. With $m_o = 1$ the following values were obtained per sample:

(a) $X_{ij} - m_o = D_{ij}$
(b) $|X_{ij} - m_o| = |D_{ij}|$
(c) $r(|X_{ij} - m_o|) = r(|D_{ij}|)$

5. The frequencies,

f_s = the number of observations with $r(|D_{ij}|) = s$ such that $D_{ij} > 0$ $(s = 1,2,\ldots,8)$

were recorded for B samples of size 8. This was done for B = 60, 80, 100, 125, and 150.

6. Then the empirical probabilities,

$$p'_s = f_s/B \qquad s = 1, 2,\ldots,8$$

were computed. The values are in Table II. The behavior of these values as the number of samples increases is shown in Figure I.

Table II

f'_s and p'_s values under H_o for B = 60,80,100,125, and 150

$r(\|D_{ij}\|)=s$	B = 60 f_s	B = 60 p'_s	B = 80 f_s	B = 80 p'_s	B = 100 f_s	B = 100 p'_s
1	34	0.5667	43	0.5375	53	0.53
2	32	0.5333	43	0.5375	59	0.59
3	33	0.5500	45	0.5625	57	0.57
4	30	0.5000	39	0.4875	48	0.48
5	28	0.4667	38	0.4750	46	0.46
6	31	0.5167	41	0.5125	48	0.48
7	35	0.5833	45	0.5625	56	0.56
8	25	0.4167	34	0.4250	43	0.43

$r(\|D_{ij}\|)=s$	B = 125 f_s	B = 125 p'_s	B = 150 f_s	B = 150 p'_s
1	64	0.512	78	0.52
2	74	0.592	86	0.5733
3	67	0.536	77	0.5133
4	65	0.52	77	0.5133
5	58	0.464	69	0.46
6	63	0.504	71	0.4733
7	75	0.60	89	0.5933
8	60	0.48	73	0.4867

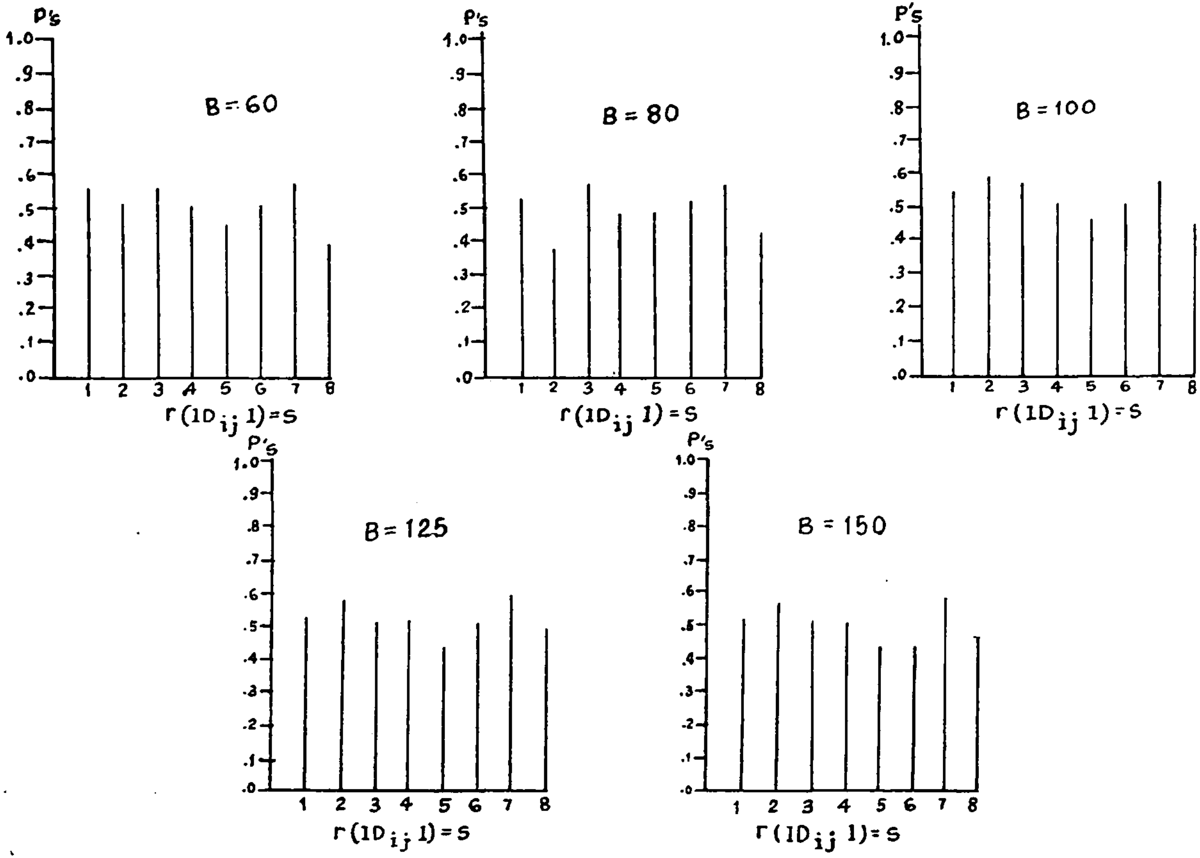

Figure 1. f's and p'_s Values Under H_o for B = 60, 80, 100, 125, and 150

7. Using the empirical p'_s for B = 150 the exact P'(I) values were computed considering the rejection regions mentioned. It can be verified from Table I that these rejection regions if no departure exists have their corresponding nominal P(I) values; in particular,

(T; T = 0,1) correspond to nominal P(I) = 0.01;
(T; T = 0,1,2,3,4,5,) correspond to nominal P(I) = 0.05;
(T; T = 0,1,2,3,4,5,6,7,8) correspond to nominal P(I) = 0.10.

To illustrate the computations involved, consider the case where the rejection region is (T; T = 0,1). Hence,

$$\text{exact } P'(I) = Pr(T = 0)_{H_o} + Pr(T = 1)_{H_o}$$

$$= \prod_{s=1}^{8} (1-p'_s) + p'_1 \prod_{s=2}^{8} (1-p'_s)$$

$$= (1-0.52)(1-0.5733)(1-0.5133)(1-0.5133)(1-0.46)(1-0.4733)(1-0.5933)(1-0.4867) + (0.52)(1-0.5733)(1-0.5133)(1-0.5133)(1-0.46)(1-0.4733)(1-0.5933)(1-0.4867)$$

$$= 0.0060.$$

The results of similar computations are given in Table III. So that the effect of the model may be seen, the exact P(I) values (i.e. with no departure considered) are also entered in this table together with the values of

$$d = \text{exact } P'(I) - \text{exact } P(I).$$

Table III

Nominal P(I)	Rejection Region	Exact P(I)	Exact P'(I)	d
0.01	0,1	0.0080	0.0060	-0.0020
0.05	0,1,2,3,4,5	0.0390	0.0332	-0.0580
0.10	0,1,2,3,4,5.6,7,8	0.0980	0.0853	-0.01270

It is seen from Table III that exact P' (I) differs from exact P(I) by approximately 10% of nominal P(I). This attests to the robustness of the WSRT under the model considered.

Actually, exact P'(I)'s are smaller than exact P(I)'s. This means that the existence of the departure has *decreased* the probability of rejecting a true hypothesis, and from this point of view, the performance of the WSRT has even *improved.*

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